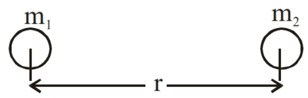
Gravitation

- Newton's law of gravitation : Newton in 1665 formulated that the force of attraction between two masses m_1 and m_2 as

$$F = \frac{Gm_1m_2}{r^2}$$



where $G = 6.67 \times 10^{-11}$. Nm^{-2} and is called universal gravitational constant.

- Gravitational field Intensity : Gravitational force per unit mass placed at a point is called gravitational field intensity at that point. Gravitational field intensity of earth is 'g'

$$\vec{I} = \frac{\vec{F}}{m}$$ where test mass m is very very small.

- Gravitational potential (V_g) : Gravitational potential at a point is the amount of work done to bring a unit mass from infinity to that point under the influence of gravitational field of a given mass M, $V_g = -\frac{GM}{r}$

- Gravitational potential and field due to system of discrete mass distribution.

$V = V_1+V_2+V_3+......$ i.e. $V = \sum_{i=1}^{N} V_i$

$\vec{I} = \vec{I}_1 + \vec{I}_2 + \vec{I}_3 + ...$ i.e. $\vec{I} = \sum_{i=1}^{N} \vec{I}_i$

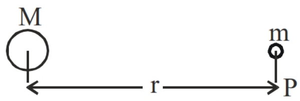
- Gravitational potential and field due to system of continuous mass distribution.

$V = \int dV$ where dV is potential due to elementary mass dM.

$\vec{I} = \int d\vec{I}$ where $d\vec{I}$ is field intensity due to elementary mass dM.

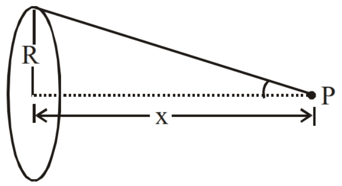
- Gravitational potential energy of two mass system is the amount of work done to bring a mass m from infinity to the point P under the influence of gravitational field of a given mass M. $U_g = -\frac{GMm}{r}$

where, U_g is G.P.E. of two mass system. Note that $U_g = mV_g$



- In general, gravitational potential energy of a system is work done against gravitational force in assembling the system from its reference configuration. Infinite mutual separation is reference configuration for mass-system.
- Gravitational field intensity due to a ring of radius R, mass M at any point on the axial line at a distance x from the centre of the ring is

$$E_g = \frac{GM.x}{(R^2 + x^2)^{3/2}}$$



The field is directed towards the centre. At the centre of the ring E_g is minimum (= 0) and E_g is maximum at

$$x = \frac{R}{\sqrt{2}}$$

- Relation between Field and potential : $I = \frac{-dV}{dr} \Rightarrow \vec{I} = \frac{-\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$

  $dV = -\vec{I}.\vec{dr}$

- Work done against gravitational force in changing the configuration of a system

  = P.E. in final configuration – P.E. in initial configuration.

  i.e. $\text{Work done} = U_2 - U_1 = W_{\text{Against gravitational force}} = -W_{\text{by gravitational force}}$

- Variation of g with height

  $$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} \text{ if } h > \frac{R}{10}$$

  $$g' = g\left(1 - \frac{2h}{R}\right) \text{ if } h < \frac{R}{10}$$

  Note g never becomes zero with height, that is, $g \to 0$ if $h \to \infty$

- Variation of g with depth (d)

  $g' = g\left(1 - \frac{d}{R}\right)$; where g is acceleration due to gravity at earth surface.

- Variation of g with rotation of earth / latitude

  $$g' = g\left(1 - \frac{R\omega^2}{g}\cos^2\lambda\right)$$

  that is, g is maximum at the poles and minimum at the equator

- Escape velocity $v_e = \sqrt{\frac{2GM}{R}}$;

  Escape velocity is the minimum velocity required to escape a mass from the surface of the earth/ planet from its gravitational. If velocity provided is greater than or equal to escape velocity, the mass will never come back to the earth/planet.

- <u>Planetry motion</u>

  Oribit velocity $v_o = \sqrt{\frac{GM}{r}}$ from the fact $\frac{GMm}{r^2} = \frac{mV^2}{r} = \text{Required Centripetal force}$

  where v_o is speed with which a planet or a satellite moves in its orbit and r is the radius of the orbit.

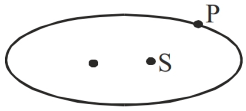
  Time period $T = \frac{2\pi r}{v_o}$ or $\boxed{T^2 = \frac{4\pi^2 r^3}{GM}}$; where v_0 = orbital velocity = $\sqrt{\frac{GM}{r}}$

  Kinetic Energy $KE = \frac{1}{2}mv_o^2 = \frac{GMm}{2r}$, Potential Energy $PE = -\frac{GMm}{r}$

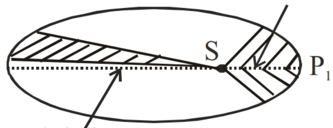
  Net energy $E = KE + PE = -\frac{GMm}{2r}$

- Kepler's Laws

  First Law : The planets revolve around the sun in the elliptical orbits with sun at one of the focus.



Second Law : The radial line sweeps out equal area in equal interval of time. This law may be derived from law of conservation of angular momentum.



Areal velocity $= \frac{dA}{dt} = = \frac{L}{2m} =$ constant

$\because$ Torque about axis of rotation is zero so angular moment is constant i.e. $I_1\omega_1 = I_2\omega_2$

$\Rightarrow (mr_1)(v_1) = (mr_2)(v_2) \Rightarrow \frac{v_1}{v_2} = \frac{r_2}{r_1}$

Thus $\frac{v_1}{v_2} = \frac{r_2}{r_1}$ or $\frac{v_{perihelion}}{v_{aphelion}} = \frac{r_{aphelion}}{r_{perihelion}}$ that is, when the planet is closer to the sun it moves fast.

Third Law: The square of the time period of a planet is proportional to he cube of a semimajor axis

$$T^2 \propto a^3 \text{ or } T^2 \propto r^3$$

If eccentricity of the orbit is e then $\frac{r_{aphelion}}{r_{perihelion}} = \frac{r_{max}}{r_{min.}} = \frac{a+ae}{a-ae} = \frac{1+e}{1-e}$

- Weightlessness in a satellite :

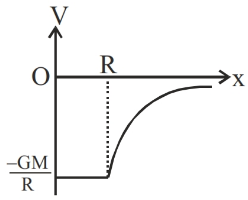
  Net force towards centre = $F_C = ma_c \Rightarrow \left(\frac{GMm}{r^2} - N\right) = m\frac{V^2}{r}$ where N is contact force by the surface

  $\Rightarrow \frac{GMm}{r^2} - N = m\left(\frac{GM}{r^2}\right)$ or $N = 0$ that is, the surface of satellite does not exert any force on the body and hence its apparent weight is zero.

- Gravitational potenial due to a ring at any point on its axis, assuming mass of the ring is uniformly or nonuniformly distributed is

$$V = \frac{-GM}{\sqrt{R^2 + x^2}} \text{ ; potential at the centre is } \frac{-GM}{R}$$

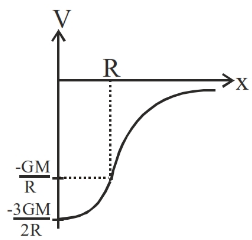
- Graviational potential due do a shell



$$V_{in} = V_{sur} = \frac{-GM}{R}; \quad V_{out} = \frac{-GM}{x}$$

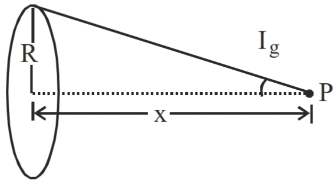
- Gravitational potential due to a solid sphere of radius R

$$V_{in} = \frac{-GM}{2R^3}(3R^2 - x^2) \quad \text{for} \quad 0 \le x \le R$$



$$V_{sur} = -\frac{GM}{R} \text{ for } x = R; \quad V_{out} = \frac{-GM}{x} \text{ for } x > R$$

- Gravitational field intensity due to a disc

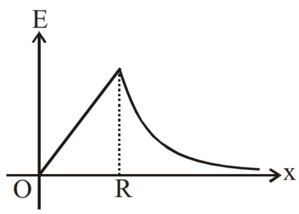


$$E=\frac{2GM}{R^2}\left[1-\frac{x}{\sqrt{x^2+R^2}}\right]=\frac{2GM}{R^2}[1-\cos\theta]$$

- Gravitational field intensity due to a solid sphere

$$E_{in}=\frac{GMx}{R^3}\quad \text{for } x<R$$

$$E_{sur}=\frac{GM}{R^2},\ E_{out}=\frac{GM}{x^2}\quad x\geq R$$



- Gravitational field intensity due to a hollow sphere

$E_{in}=0\ ;\ x<R$

$$E_{surface}=\frac{GM}{R^2}\ ;x=R$$

$$E_{out}=\frac{GM}{x^2};\ x>R$$